Math 601 Midterm 1 Solution

This is the solution sheet for Midterm 1 of Math 601. Watch out for typos/errors. Please let me know if you spot any of them.

Question 1. (18 pts)

Determine whether each of the following statements is true or false. You do NOT need to explain.

(a) Let V be a linear subspace of $\mathbb{R}^n$. We have vectors $v_1, \cdots, v_k$ and $w_1, \cdots, w_\ell$ in V. Suppose $v_1, \cdots, v_k$ are linearly independent, and $w_1, \cdots, w_\ell$ span V. Then $k \leq \ell$.

(b) Let U and W be subspaces of the vector space V. If $U \subseteq W$, then $U + W = W$.

(c) An $(n \times n)$ matrix is invertible if and only if it is row equivalent to the $(n \times n)$ identity matrix I_n.

(d) Suppose A is an $(n \times m)$ matrix, then $\dim \operatorname{Im}(A) \leq n$ and $\dim \ker(A) \leq m$.

(e) Let A, B, C be three $(n \times n)$ square matrices. If $AB = AC$, then $B = C$.

(f) The linear system

$$\begin{array}{rcrcrcl} x & + & 10y & - & 3z & = & 3 \\ 3x & + & 4y & + & 9z & = & 1 \\ 2x & + & 5y & - & 2z & = & 8 \end{array}$$

has exactly two solutions.

Solution:

(a) True

(b) True

(c) True

(d) True

(e) False

(f) False (a linear system can never have precisely two solutions.)

Question 2. (10 pts)

A line L in $\mathbb{R}^3$ passes through the point $(0, 1, 0)$. Suppose L is parallel to the plane $x + y + z = 0$ and is orthogonal to the line

$$x = 2t, y = t + 1, z = -t.$$

Find parametric equations of the line L.

Solution: The direction vector of L is orthogonal to both the normal vector $u = (1, 1, 1)$ of the plane and the direction vector $v = (2, 1, -1)$ of the other line.

Calculate the cross product of u and v:

$$u \times v = (-2, -3, -1)$$

So we have the following parametric equations of L:

$$\begin{cases} x = -2t \\ y = 3t + 1 \\ z = -t \end{cases}$$

Question 3. (15 pts)

Given

$$A = \begin{bmatrix} 2 & 2 & -3 & 1 & 13 \\ 1 & 1 & 1 & 1 & -1 \\ 3 & 3 & -5 & 0 & 14 \\ 6 & 6 & -2 & 4 & 16 \end{bmatrix}$$

(a) Find a basis of Ker(A).

Solution: First, use elementary row operations to get the reduced row echelon form of A.

$$\text{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So all elements in Ker$A$ are of the form

$$t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ 4 \\ -5 \\ 1 \end{bmatrix}$$

So

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 0 \\ 4 \\ -5 \\ 1 \end{bmatrix}$$

form a basis of the kernel.

(b) Find a basis of the row space of A.

Solution: The three nonzero rows in the reduced row echelon form of A form a basis of the row space of A. That is

$$u_1 = (1, 1, 1, 1, -1)$$

$$u_2 = (0, 0, -5, -1, 15)$$

$$u_3 = (0, 0, 0, 1, 5)$$

form a basis of the row space of A.

(c) Find a basis of $\mathrm{Im}(A)$.

Solution: Use rref(A) from the part (a), we see that the 1st, 3rd and 4th columns of A form a basis of $\mathrm{Im}(A)$. That is,

$$w_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 6 \end{bmatrix}, w_2 = \begin{bmatrix} -3 \\ 1 \\ -5 \\ -2 \end{bmatrix}, w_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 4 \end{bmatrix}$$

form a basis of $\mathrm{Im}(A)$.

Question 4. (10 pts)

Let $M_2(\mathbb{R})$ be the space of all (2×2) matrices with real coefficients. The set

$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

is a basis of $M_2(\mathbb{R})$. Find the coordinates of $A = \begin{pmatrix} 5 & 3 \\ 3 & 1 \end{pmatrix}$ with respect to the basis S.

Solution: We need to write

$$\begin{pmatrix} 5 & 3 \\ 3 & 1 \end{pmatrix} = a_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + a_4 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

That is, we need to solve the linear system

$$\begin{cases} a_1 + a_2 + a_3 + a_4 = 5 \\ a_1 - a_2 - a_3 = 3 \\ a_1 + a_2 = 3 \\ a_1 = 1 \end{cases}$$

Simply use back substitution. We have

$$a_1 = 1, a_2 = 2, a_3 = -4, a_4 = 6$$

So

$$[A]_S = \begin{bmatrix} 1 \\ 2 \\ -4 \\ 6 \end{bmatrix}$$

Question 5. (10 pts)

Determine whether $x = \begin{bmatrix} 4 \\ 5 \\ 6 \\ -1 \end{bmatrix}$ lies in the linear span of the vectors

$$v_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 2 \end{bmatrix} \text{ and } v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 3 \end{bmatrix}.$$

Solution: Write down the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 3 & 4 & -2 & 5 \\ 2 & -1 & 1 & 6 \\ 5 & 2 & 3 & -1 \end{array}\right]$$

by applying elementary row operations, we get

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 5/3 \\ 0 & 0 & 0 & 55/12 \end{array}\right]$$

This is inconsistent. So no solution. In other words, x is not in the linear span of v_1, v_2 and v_3.

Question 6. (6 pts)

Let V be a vector space. Suppose $F : V \to V$ is a linear transformation. Show that the kernel of F is a subspace of V.

Solution:

(1) $0 \in \ker F$, since $F(0) = 0$.

(2) If $v, w \in \ker F$, then

$$F(v + w) = F(v) + F(w) = 0 + 0 = 0.$$

So $v + w \in \ker F$.

(3) If $w \in \ker F$ and $k \in \mathbb{R}$, then

$$F(kw) = kF(w) = k \cdot 0 = 0.$$

So $kw \in \ker F$.

Therefore $\ker F$ is a subspace of V.

Question 7. (10 pts)

Show that $(t-1)$, $(t+1)$ and $(t-1)^2$ form a basis of $\mathbb{P}_2(t)$, where $\mathbb{P}_2(t)$ is the space of all polynomials of degree ≤ 2.

Solution: There are various ways to solve this problem.

(1) First method: prove that $(t-1)$, $(t+1)$ and $(t-1)^2$ are linearly independent and span $\mathbb{P}_2(t)$.

(2) Second method: prove that $(t-1)$, $(t+1)$ and $(t-1)^2$ are linearly independent and use the fact $\dim \mathbb{P}_2(t) = 3$.

(3) third method: prove that $(t-1)$, $(t+1)$ and $(t-1)^2$ span $\mathbb{P}_2(t)$ and use the fact $\dim \mathbb{P}_2(t) = 3$.

Let us the second method. Consider a linear combination of $(t-1)$, $(t+1)$ and $(t-1)^2$ such that

$$a_1(t-1) + a_2(t+1) + a_3(t-1)^2 = 0.$$

Then we want to show that $a_1 = a_2 = a_3 = 0$ is the unique solution. This would imply that $(t-1)$, $(t+1)$ and $(t-1)^2$ are linearly independent.

Regroup the coefficients, and we have the following linear system:

$$\begin{cases} -a_1 + a_2 + a_3 = 0 \\ a_1 + a_2 - 2a_3 = 0 \\ a_3 = 0 \end{cases}$$

Solve this and indeed we have the unique solution $a_1 = a_2 = a_3 = 0$. So $(t-1)$, $(t+1)$ and $(t-1)^2$ are linearly independent.

Now we know that $\dim \mathbb{P}_2(t) = 3$. Then any 3 linearly independent vectors of $\mathbb{P}_2(t)$ form a basis. Therefore $(t-1)$, $(t+1)$ and $(t-1)^2$ form a basis.

Question 8. (5 pts)

Suppose A and B are invertible $(n \times n)$ matrices. Then we know that AB is also invertible. Use this fact and the definition of the inverse of an invertible matrix to show that

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Solution: Notice that

$$B^{-1}A^{-1}(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = I$$

$$(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = A^{-1}IA = I$$

So by definition the inverse of AB, which is denoted by $(AB)^{-1}$, is $B^{-1}A^{-1}$. That is,

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Question 9. (16 pts)

Recall that two vectors $v, w \in \mathbb{R}^n$ are said to be orthogonal if their dot product is zero, that is, $v \cdot w = 0$.

(a) Let $u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} +1 \\ 0 \\ -1 \end{bmatrix}$ and $u_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ in $\mathbb{R}^3$. Determine whether u_1, u_2 and u_3 are mutually orthogonal.

Solution: A straightforward calculation shows that u_1, u_2 and u_3 are mutually orthogonal.

(b) Suppose v and w are both nonzero vectors in $\mathbb{R}^n$. Show that if $v \cdot w = 0$, then v and w are linearly independent. (Hint: For any linear combination of v and w, take the dot product of this linear combination with v (respectively w). What do you see?)

Solution: Suppose we have a linear combination

$$a_1 v + a_2 w = 0.$$

Then we need to show that $a_1 = a_2 = 0$.
Indeed, consider the dot product

$$0 = (a_1 v + a_2 w) \cdot v = a_1 \|v\|^2 + 0 = a_1 \|v\|^2$$

But $\|v\| \neq 0$. Therefore $a_1 = 0$.
Similarly,

$$0 = (a_1 v + a_2 w) \cdot w = 0 + a_2 \|w\|^2 + 0 = a_2 \|w\|^2$$

shows that $a_2 = 0$.
So v and w are linearly independent.

(c) Now suppose nonzero vectors v_1, v_2 and v_3 are mutually orthogonal in $\mathbb{R}^n$. Show that the set $\{v_1, v_2, v_3\}$ is linearly independent. (Hint: the same idea from part (b) applies.)

Solution: Suppose we have a linear combination

$$a_1v_1 + a_2v_2 + a_3v_3 = 0.$$

Then we need to show that $a_1 = a_2 == a_3 = 0$.
Consider the dot product

$$0 = (a_1v_1 + a_2v_2 + a_3v_3) \cdot v_1 = a_1\|v_1\|^2$$

But $\|v_1\| \neq 0$. Therefore $a_2 = 0$.
Similarly, by taking the dot product of $a_1v_1 + a_2v_2 + a_3v_3$ with v_2 and v_3 respectively, we have $a_2 = a_3 = 0$.
Therefore, the set $\{v_1, v_2, v_3\}$ is linearly independent.

(d) Use either the previous parts or your other favorite method to determine whether $u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} +1 \\ 0 \\ -1 \end{bmatrix}$ and $u_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ form a basis of $\mathbb{R}^3$.

Solution: From part (a), we know that u_1u_2 and u_3 are mutually orthogonal and are nonzero vectors.
By part (c), we know u_1, u_2 and u_3 are linearly independent.
Since $\dim \mathbb{R}^3 = 3$, we see that u_1, u_2 and u_3 form a basis of $\mathbb{R}^3$.